Quantum localization in open chaotic systems

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(Received 5 February 2008; revised manuscript received 16 June 2008; published 3 September 2008)

We study a quasibound state of a δ-kicked rotor with absorbing boundaries focusing on the nature of the dynamical localization in open quantum systems. The localization lengths ξ of lossy quasibound states located near the absorbing boundaries decrease as they approach the boundary while the corresponding decay rates Γ are dramatically enhanced. We find the relation $\xi \sim \Gamma^{-1/2}$ and explain it based upon the finite time diffusion, which can also be applied to a random unitary operator model. We conjecture that this idea is valid for the system exhibiting both the diffusion in classical dynamics and the exponential localization in quantum mechanics.

DOI: 10.1103/PhysRevE.78.037201 PACS number(s): 05.45.Mt, 05.60.−k, 37.10.−x

Quantum localization (QL) is one of the fascinating phenomena which cannot be expected in classical mechanics. It has been observed in various physical situations: Anderson localization (AL) in disordered systems [1], dynamical localization (DL) in chaotic systems [2], weak localization in dirty metals [3], and so on. The QL mainly originates from the interference among waves returning their initial condition. Even though classical statistical mechanics predicts that a particle in a random potential exhibits stochastic motion and thus gives rise to simple diffusion, the aforementioned interference effect stops diffusion and localizes it within some characteristic length scale referred to as the localization length, ξ. Due to the interference nature of the QL, the coherence of the wave plays an indispensable role.

The DL is a dynamical version of the AL in the sense that the distribution in momentum space stops diffusing and exhibits typical exponential localization. The DL was theoretically found in the quantum δ-kicked rotor (DKR) [2,4], a paradigm of quantum chaos [5], and experimentally realized by using ultracold atoms [6]. In the classical DKR the mean square deviation of momentum indefinitely increases linearly in time, while in quantum mechanics it follows classical evolution only for a short time. At the characteristic time scale, namely *a break time*, it then begins to saturate so that eventually the quantum diffusion is completely suppressed. The formal equivalence between the DKR and the Anderson model has been demonstrated [7].

In principle every *real* quantum system is coupled to the environment since no information can be extracted from completely closed systems. Thus it is a natural question to ask how the coupling of the quantum system to the environment modifies genuine quantum effects such as the QL. Recently, the interest in open quantum systems has been rapidly growing in the quantum chaos community [8–15]. In the semiclassical limit the openness introduces two major modifications onto the chaotic quantum systems: (i) the fractal repellers manifest themselves in quasieigenstates [10] and (ii) the density of states follows the so-called *fractal* Weyl's law [12]. The severe deviation from the random matrix theory has also been reported in chaotic scattering when the dwell time of an incident particle is extremely short (shorter than the Ehrenfest time) [13,16].

Even far from the semiclassical limit the openness of complicated quantum systems has been an interesting issue, for example, the characteristics of lasing modes in chaotic microcavities [17–20] (see [21,22] for reviews) and localization of light in random media [23–27]. Here we investigate the *open* DKR in the quantum mechanical regime focusing on the characteristics of localization of lossy modes located near the open boundary. We found that quasieigenstates of the open DKR are separated into two kinds: one is the localized state whose localization length is almost equivalent to that of the DKR *without* the absorbing boundaries and the decay rate is determined from simple overlap argument discussed below. The other is a highly lossy mode located near the boundary, which is more strongly localized and of which decay rate is determined by considering the finite time classical diffusion. We also show that all these observations and explanations are applicable to an *open* random unitary operator model.

The Hamiltonian of the DKR is given as

$$H = \frac{p^2}{2} + k \sin x \sum_n \delta(t - nT), \tag{1}$$

where k is the kick strength, and T is the time interval between successive kicks. We fix $k=14$ and $T=1$, i.e., $K=kT=14$ implying the classical dynamics is fully chaotic. Note that the semiclassical limit implies $k \to \infty$ and $T \to 0$ with kT kept constant, so that we do not consider the semiclassical regime. The time evolution of the *open* quantum DKR with absorbing boundaries [28] is described by $|\psi(T)\rangle = \hat{P}\hat{U}|\psi(0)\rangle$, where $\hat{U}$ is a unitary time evolution operator for one period without absorption, and the operator $\hat{P}$ projects the wave function to the states satisfying $|p| \leq p_c$, where p_c represents the absorbing boundary. We choose $p_c=1000$ and set $\hbar=1$. This model has been extensively studied in the context of the fidelity decay or the Loschmidt echo [29].

The DL manifests itself via the exponentially localized Floquet eigenstate of $\hat{U}$. In the *open* DKR the quasibound state (QBS) can be analogously defined as

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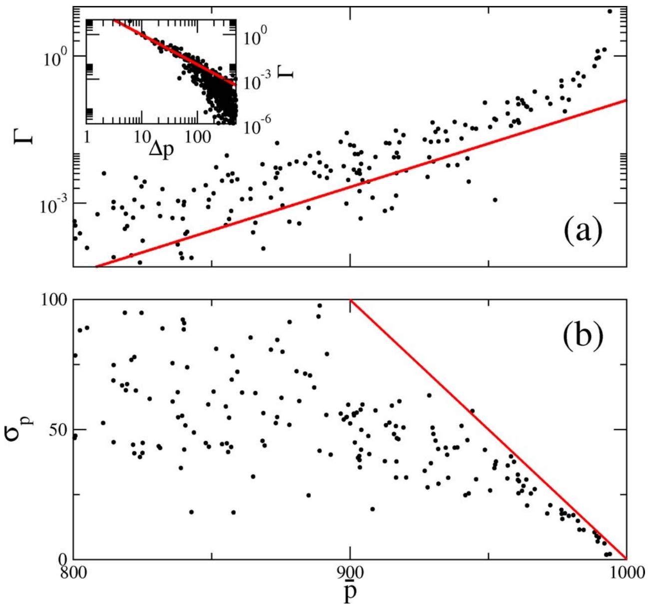

FIG. 1. (Color online) (a) The decay constants Γ of each QBS of the open DKR as a function of the average momentum $\bar{p}$ in log scale. The straight line represents Eq. (3). The inset shows Γ versus $\Delta p \equiv p_c - \bar{p}$ in log-log scale. The straight line represents Eq. (4). (b) The standard deviations σ_p as a function of $\bar{p}$. The known theory for the DKR without absorbing boundary predicts $\sigma_p \sim 50$. The straight line represents Eq. (5).

$$\hat{P}\hat{U}|\phi\rangle = e^{-\Gamma} e^{i\alpha}|\phi\rangle. \quad (2)$$

Due to the openness of the system the Floquet eigenvalues no longer lie at a unit circle in a complex plane ($\Gamma > 0$). Each QBS can be identified by its average momentum, $\bar{p} \equiv \langle\phi|\hat{p}|\phi\rangle$. In one period T the survival probability of the given QBS is given as $e^{-2\Gamma}$ so that the lifetime of the state, τ_L, is obtained by $\tau_L \propto \Gamma^{-1}$.

In Fig. 1 we present the decay constant Γ and the width of the momentum distribution σ_p of QBSs labeled by $\bar{p}$. Here two visible features are clearly observed: (i) For $\bar{p}$ much smaller than p_c, Γ increases exponentially and σ_p remains constant with some fluctuation. (ii) As $\bar{p}$ approaches p_c, Γ more rapidly increases and σ_p linearly decreases.

The first feature can be easily understood. For $\bar{p} \ll p_c$ the localized state has negligible influence from the absorbing boundary so that the QBS is not so different from the original Floquet state of the DKR *without* absorbing boundaries as shown in Fig. 2(a). This is also true even quantitatively since it is fairly good to estimate σ_p based upon the well-known relation $\sigma_p \sim \xi \sim D/2 \sim K^2/4$ [30], where D is the classical diffusion constant. It implies that the QBS can be described as $\phi \sim \exp(-|p-\bar{p}|/\xi)$. The decay constant Γ is then determined by considering the overlap between the exponential tail of the localized QBS and the absorbing region given as $|p| \geq p_c$: $1 - e^{-2\Gamma} \sim \xi^{-1}\int_{p_c}^{\infty} \exp(-2|p-\bar{p}|/\xi)dp$. It leads us to

$$\Gamma \sim \exp\left[-\frac{2}{\xi}\Delta p\right], \quad (3)$$

where $\Delta p = p_c - \bar{p}$. Figure 1(a) clearly shows this expectation is correct.

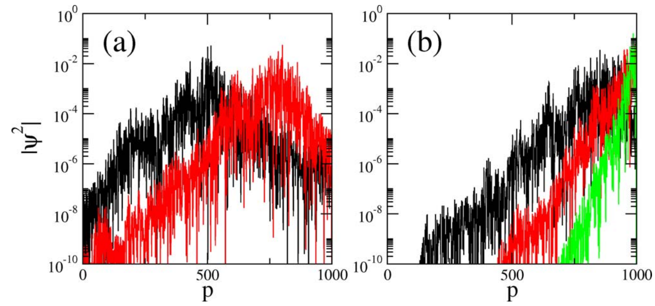

FIG. 2. (Color online) (a) The momentum distributions of two typical exponentially localized QBSs with $\bar{p} \sim 502.86$ and 765.47, where $\tau_L > \tau_B$ is satisfied. (b) As $\bar{p}$ approaches p_c through 865.80, 961.04, and 982.88 (from the left to the right), the distribution becomes narrower.

As $\bar{p}$ approaches p_c, however, the absorbing boundary has a dramatic influence on the QBSs. They become much more lossy and even more strongly localized, which is unlikely because the strongly localized mode has smaller overlap with the absorbing region, i.e., becomes less lossy. It is worth mentioning that here we exploit the absorbing boundaries to open the system to the environment. Usually the coupling to the environment introduces dissipation or decoherence, which destroys the coherence itself, let alone the localization. Therefore, one expects that the localization length increases. In this sense the absorbing boundaries are special. The absorbing boundary condition has been widely used and relevant for many physical situations, e.g., optical microcavites, ionization processes, random lasers, and so on.

In a usual DKR there is only one important time scale [31], namely the break time τ_B, at which the diffusion stops so as to determine the localization of the Floquet state: $\sigma_p{}^2 \sim \xi^2 \sim D\tau_B$. In an *open* DKR, however, we should consider one more time scale, the lifetime, τ_L. A clue comes from the fact that the crossover from constant to decreasing $\sigma_p(\bar{p})$ takes place around $\tau_L \sim \tau_B$.

The system undergoes its meaningful dynamics only for $t < \tau_L$ in the sense that the probability distribution no longer changes except for overall decaying. If $\tau_L < \tau_B$ is satisfied, therefore, the meaningful dynamics stops before the break time is reached. It means that the classical diffusion plays a dominant role in the decay process since a particle disappears before the quantum suppression of classical diffusion takes place. In other words, the particle escapes when it diffusively arrives at the absorbing boundary. One can expect that the lifetime τ_L of a given QBS with $\bar{p}$ is determined simply from the duration time for which the particle travels from $\bar{p}$ to p_c by diffusion: $\tau_L \sim (p_c - \bar{p})^2/D$. This can be rewritten as

$$\Gamma \sim \frac{D}{\Delta p^2}, \quad (4)$$

which works quite well as shown in the inset of Fig. 1(a). It is emphasized that Eq. (4) is unexpected because we are not dealing with the semiclassical regime. The classical diffusive dynamics decides the decay constant even in the deep quantum regime when the loss is large enough.

Considering the above argument the linear decrease of σ_p can also be understood. The meaningful dynamics stops at τ_L ($<\tau_B$) so that σ_p^2 is determined not from $D\tau_B$ but from $D\tau_L$. By using Eq. (4) we obtain

$$\sigma_p \sim \sqrt{\frac{D}{\Gamma}} \sim \Delta p, \quad (5)$$

which is clearly seen in Fig. 1(b).

In some sense very lossy modes near the absorbing boundary are not interesting since they decay so fast. They do not contribute to long time dynamics so that they form only broad peaks even in the scattering cross section. Sometimes such a mode, nevertheless, becomes of great importance; for example, a very lossy mode can play a dominant role in lasing operation, where an external energy input compensates for the loss of the mode [18]. It is also worth mentioning that the spatial shape of an individual mode is recently measured in the experiment of light in a random medium, where the lossy mode strongly localized near the boundary is also observed [25,26].

The main theme of our work is that the lossy QBS of the open DKR near the absorbing boundary shows rather stronger localization. In principle this is applicable to any system that exhibits diffusion in classical mechanics and exponential localization in quantum mechanics. We consider one more example originating from the one-dimensional Anderson model. Instead of Anderson's tight-binding Hamiltonian we exploit the so-called random unitary operator (RUO) $\hat{U}=\hat{D}\hat{S}$, where $\hat{D}_{mn}=e^{i\theta_m}\delta_{mn}$ with a random phase θ_m [32], and an infinite dimensional matrix $\hat{S}$ is defined as

$$\hat{S} = \begin{pmatrix} \ddots & rt & -t^2 & & & \\ & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -tr & r^2 & -rt & \\ & & & rt & r^2 & \\ & & & -t^2 & -tr & \ddots \end{pmatrix}. \quad (6)$$

Here, $\hat{S}$ describes the scattering process via the relation $\phi_i'=\hat{S}_{ij}\phi_j$, where ϕ (ϕ') represents the incoming (outgoing) flux for a scatterer. In the one-dimensional case two fluxes with the opposite direction at each site between two neighboring scatterers form two adjacent vector components, ϕ_{2k} and ϕ_{2k+1} [32]. The condition $r^2+t^2=1$ holds to ensure unitarity. For all calculation we fix $t=0.4$. In order to control the width of the off-diagonal components, which roughly corresponds to the parameter K of a DKR, we simply multiply $\hat{S}$: i.e., $\hat{U}_n=\hat{D}\hat{S}^n$ ($n=1,2,3,\ldots$). Note that the parameter K of a classical DKR determines the maximum momentum transfer to a particle from each kick. The nonzero (i,j)th off-diagonal component of $\hat{U}$ implies there exists a nonzero transition probability between these two states: roughly speaking $K \sim \max(i-j)$ satisfying $\langle i|\hat{U}|j\rangle \neq 0$. One then expects that the larger n the bigger the corresponding *effective* K. We now introduce the projection operator $\hat{P}$ and the QBSs of $\hat{P}\hat{U}_n$ are investigated in a similar way.

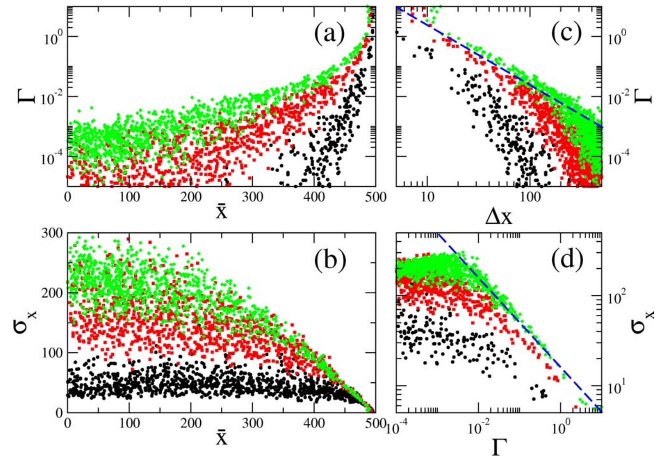

FIG. 3. (Color online) From bottom to top $n=10,20,30$ are exploited for all figures. (a),(b) The same as Figs. 1(a) and 1(b), respectively, except that a RUO model is considered. (c) Γ versus Δx in log-log scale. The straight line represents Eq. (4). (d) σ_x versus Γ in log-log scale. The straight line represents Eq. (5).

In Figs. 3(a) and 3(b), we present the decay constant Γ and the standard deviation σ_x, respectively, of an individual QBS of $\hat{P}\hat{U}_n$ for various n. Once again two visible features are observed. For $\bar{x}$ ($\equiv\langle\phi|\hat{x}|\phi\rangle$) much smaller than x_c (x is used here instead of p), as $\bar{x}$ increases, Γ exponentially increases and σ_x remains constant. As $\bar{x}$ closely approaches x_c, however, Γ more rapidly increases and σ decreases linearly. These observations are exactly analogous to those obtained in the open DKR (see Fig. 1). As n increases, the crossover from constant to linearly decreasing σ_x is reduced since the bigger n is the larger the effective K, consequently the longer the break time. Figures 3(c) and 3(d) reconfirm the relations (4) and (5), respectively, i.e., $\Gamma \propto \Delta x^{-2}$ and $\sigma \propto \Gamma^{-1/2}$. It is emphasized that these results are independent of n.

A final remark is in order. We have shown that the results obtained in the DKR can be directly applied to those of the RUO model. It is noted, however, that chaos is different from random motion. The chaotic dynamics has more structures in phase space, e.g., stable and unstable manifolds associated with periodic orbits. The strong localization on the unstable manifolds or the strange repellers in chaotic open systems has been reported [10,22]. One might expect that rather

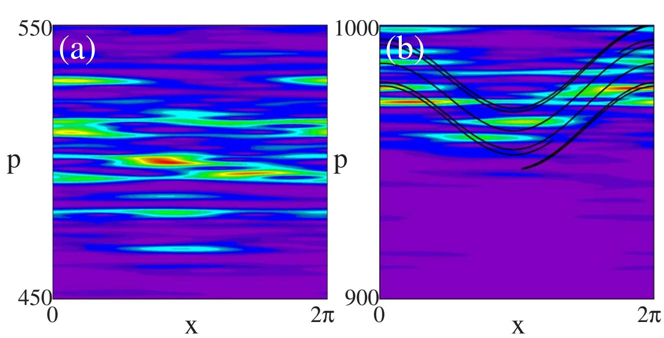

FIG. 4. (Color online) Husimi distribution functions of the QBS for (a) $\bar{p}\simeq 502$ and (b) $\bar{p}\simeq 972$. The black curve represents the classical unstable manifold.

stronger localization observed here, i.e., the reduced localization length, may be due to such underlying classical structures in phase space. In fact, it is shown in Fig. 4(b) that the quantum distribution of the lossy QBS is localized along the unstable manifolds (the effect is not so dramatic because the situation is far from the semiclassical limit), while most other QBSs exhibit rather uniform distribution in phase space [a typical example is shown in Fig. 4(a)]. It is emphasized that we also observe the similar strong localization in the RUO model, where neither dynamical chaos nor manifoldlike structure exists. It implies that the strong localization of the lossy QBS observed in the DKR is ascribed not to the manifold structures but to the finite time diffusion.

In summary, we have shown that the QBSs with their lifetime smaller than the break time, i.e., $\tau_L < \tau_B$, exhibit rather stronger localization and considerable loss. In this case, the main mechanism of the decay is determined from classical diffusion, which gives the relation (4). In addition, before the break time is reached the classical diffusion effectively stops so that the state with much narrower momentum distribution is formed. The width of the distribution is then described as Eq. (5). Such a simple explanation can also be successfully applied to a random unitary operator model. We believe that our theory is valid for various physical situation that the diffusion takes place in classical mechanics and strong exponential localization exists in quantum mechanics. We hope that our expectation is experimentally proven, e.g., by direct observation of geometrical shapes of quasibound states in light in random media.

This work was supported by Korea Research Foundation Grant No. KRF-2006-312-C00543 and by Korea Science and Engineering Foundation Grant No. R01-2005-000-10678-0.

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958); Rev. Mod. Phys. **50**, 191 (1978).

[2] G. Casati, B. V. Chirikov, F. M. Izraelev, and J. Ford, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, edited by G. Casati and J. Ford, Lecture Notes in Physics Vol. 93 (Springer, Berlin, 1979).

[3] G. Bergmann, Phys. Rep. **107**, 1 (1984).

[4] F. M. Izraelev, Phys. Rep. **196**, 299 (1990).

[5] L. E. Reichl, *The Transition to Chaos in Conservative Classical Systems: Quantum Manifestations* (Springer, New York, 1992).

[6] F. L. Moore, J. C. Robinson, C. F. Bharucha, B. Sundaram, and M. G. Raizen, Phys. Rev. Lett. **75**, 4598 (1995).

[7] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).

[8] F. Borgonovi, I. Guarneri, and L. Rebuzzini, Phys. Rev. Lett. **72**, 1463 (1994).

[9] G. Casati, G. Maspero, and D. L. Shepelyansky, Phys. Rev. E **56**, R6233 (1997).

[10] G. Casati, G. Maspero, and D. L. Shepelyansky, Physica D **131**, 311 (1999).

[11] G. Benenti, G. Casati, I. Guarneri, and M. Terraneo, Phys. Rev. Lett. **87**, 014101 (2001).

[12] W. T. Lu, S. Sridhar, and M. Zworski, Phys. Rev. Lett. **91**, 154101 (2003).

[13] H. Schomerus and J. Tworzydlo, Phys. Rev. Lett. **93**, 154102 (2004); H. Schomerus and P. Jacquod, J. Phys. A **38**, 10663 (2005).

[14] T. Kottos, J. Phys. A **38**, 10761 (2005).

[15] J. P. Keating, M. Novaes, S. D. Prado, and M. Sieber, Phys. Rev. Lett. **97**, 150406 (2006).

[16] S. Rahav and P. W. Brouwer, Phys. Rev. Lett. **95**, 056806 (2005); **96**, 196804 (2006).

[17] J. U. Nöckel and A. D. Stone, Nature (London) **385**, 45 (1997).

[18] C. Gmachl *et al.*, Science **280**, 1556 (1998).

[19] S.-B. Lee *et al.*, Phys. Rev. Lett. **88**, 033903 (2002).

[20] T. Harayama, T. Fukushima, S. Sunada, and K. S. Ikeda, Phys. Rev. Lett. **91**, 073903 (2003).

[21] A. D. Stone, Phys. Scr., T **T90**, 248 (2003).

[22] S. W. Kim *et al.*, Prog. Theor. Phys. Suppl. **166**, 119 (2007).

[23] D. S. Wiersma, P. Bartolini, A. Lagendijk, and R. Righini, Nature (London) **390**, 671 (1997).

[24] A. A. Chabanov, M. Stoytchev, and A. Z. Genack, Nature (London) **404**, 850 (2000).

[25] P. Sebbah, B. Hu, J. M. Klosner, and A. Z. Genack, Phys. Rev. Lett. **96**, 183902 (2006).

[26] K. Yu. Bliokh *et al.*, Phys. Rev. Lett. **97**, 243904 (2006).

[27] S. E. Skipetrov and B. A. van Tiggelen, Phys. Rev. Lett. **96**, 043902 (2006).

[28] F. Borgonovi, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. A **43**, 4517 (1991).

[29] G. Casati, G. Maspero, and D. L. Shepelyansky, Phys. Rev. Lett. **82**, 524 (1999).

[30] D. L. Shepelyansky, Phys. Rev. Lett. **56**, 677 (1986).

[31] In fact there is one more important time scale, which is the Ehrenfest time τ_E. In our case one can safely ignore it since it is considerably small, namely, $\tau_E \sim 1$.

[32] I. A. Koshovets, Theor. Math. Phys. **89**, 1249 (1991); O. Bourget, J. S. Howland, and A. Joye, Commun. Math. Phys. **234**, 191 (2003); A. Joye, Ann. Henri Poincare **5**, 347 (2004); E. Hamza, A. Joye, and G. Stolz, Lett. Math. Phys. **75**, 255 (2006).